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Revisiting Critical Vortices in Three-Dimensional SQED

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Abstract

We consider renormalization of the central charge and the mass of the $\mathcal{N} = 2$ supersymmetric Abelian vortices in $2 + 1$ dimensions. We obtain $\mathcal{N} = 2$ supersymmetric theory in $2 + 1$ dimensions by dimensionally reducing the $\mathcal{N} = 1$ SQED in $3 + 1$ dimensions with two chiral fields carrying opposite charges. Then we introduce a mass for one of the matter multiplets without breaking $\mathcal{N} = 2$ supersymmetry. This massive multiplet is viewed as a regulator in the large mass limit. We show that the mass and the central charge of the vortex get the same nonvanishing quantum corrections, which preserves BPS saturation at the quantum level. Comparison with the operator form of the central extension exhibits fractionalization of a global $U(1)$ charge; it becomes $\pm 1/2$ for the minimal vortex. The very fact of the mass and charge renormalization is due to a “reflection” of an unbalanced number of the fermion and boson zero modes on the vortex in the regulator sector.

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1 Introduction

$\mathcal{N} = 2$ supersymmetric QED with the Fayet–Iliopoulos term in $2 + 1$ dimensions supports Abrikosov–Nielsen–Olesen (ANO) vortices [1, 2]. These classical solutions are 1/2-BPS saturated (two out of four supercharges are conserved). Quantum corrections to the vortex mass and central charge were discussed in the literature more than once. It is firmly established [3] that there are two fermion zero modes on the vortex implying that the supermultiplet to which the vortex belongs is two-dimensional. This is a short supermultiplet. Hence, the classical BPS saturation cannot be lost in loops.

Particular implementation of the vortex BPS saturation turned out to be a contentious issue, almost to the same extent as it had happened with two-dimensional kinks in $\mathcal{N} = 1$ models (for reviews see [4], Sect. 3.1 in [5], and [6]). The authors of [1] and [7] obtained a vanishing quantum correction to the vortex mass using the following eigenvalue densities:

$$n_B(w) - n_F(w) \propto \delta(w), \quad (1)$$

where $n_{B(F)}$ is the bosonic (fermionic) density of states. The vanishing mass correction ensues since

$$\Delta M_v \propto \int dw (n_B(w) - n_F(w)) w = 0. \quad (2)$$

Since the vortex mass M_v is proportional to the Fayet–Iliopoulos (FI) parameter ξ , and ξ is renormalized in one loop, the above result caused a problem.

Later new calculations of the vortex mass were undertaken and a non-vanishing one-loop correction to the vortex mass was reported in [8, 9]. It was shown [3] that the central charge also gets a correction, so that the BPS saturation of the vortex persists at the one-loop level. However, the (dimensional) regularization that was used in the most detailed paper [3], expressly written to discuss three-dimensional supersymmetric vortices, does not allow one to treat in a straightforward manner the Chern–Simons (CS) term, whose role in the problem at hand is important. In this paper we use another regularization method in which the CS term naturally appears in the limit of large regulator mass. This mass is also crucial in the operator form of the centrally extended algebra which we derive at one loop. Our operator expression for the central extension includes the Noether charge (20).

Here we would like to close these gaps. In this paper we revisit the issue using a physically motivated regularization which is absolutely transparent.

We recalculate the renormalization of the vortex mass at one loop

$$M_{v,R} = 2\pi \left(\xi_R - \frac{m}{4\pi} \right) \quad (3)$$

and the one-loop effect in the central charge. (Here ξ_R is the renormalized value of the FI parameter, m is the matter field mass,

$$m = e\sqrt{2\xi_R},$$

and the subscript R stands for renormalized.) The above result is in agreement with the previous calculations [3, 8]. Needless to say, our direct calculation confirms BPS saturation, $M_{v,R} = |Z_R|$. Moreover, it demonstrates that, in the limit of the large regulator mass, regulator's role is taken over by the Chern–Simons term. A new finding obtained by comparing the central charge calculation with the operator form of the central extension is a U(1) global charge fractionalization. The operator expression for the central extension which we derived in our regularization is presented in Eqs. (19) and (20). Then we discuss the central charge/vortex mass renormalization to all orders in perturbation theory, see Eq. (55).

$\mathcal{N} = 2$ SQED Lagrangian in $2 + 1$ dimensions (four supercharges) can be obtained by dimensional reduction of $\mathcal{N} = 1$ supersymmetric Lagrangian in $3 + 1$ dimensions. In order to have a well defined anomaly-free SQED in four dimensions, one has to have two matter superfields, say Φ and $\tilde{\Phi}$, with the opposite charges. Since there is no chirality in three dimensions, in three-dimensional SQED, in principle, it is sufficient to keep a single superfield (say, Φ), while $\tilde{\Phi}$ can be eliminated. This is a minimal setup which is routinely considered. The four-dimensional anomaly is reflected in three dimensions in the form of a “parity anomaly” [10, 11] and the emergence of the Chern–Simons term, as will be explained momentarily.

When we speak of eliminating $\tilde{\Phi}$ we should be careful. Eliminating does not mean discarding. As was briefly discussed in [5] (Sect. 3.2), a perfectly safe method of getting rid of $\tilde{\Phi}$ is to make the tilded fields heavy. Then the corresponding supermultiplet decouples and does not appear in the low-energy theory. It leaves a trace, however, in the form of the Chern–Simons term [10, 11], as shown in Sect. 4.

There is a well-known method of making the tilded fields heavy without altering the masses of the untilded fields. It works in three dimensions. One can introduce a “real” mass \tilde{m} [12] (a three-dimensional analog of the

twisted mass in two dimensions [13]) without breaking $\mathcal{N} = 2$ supersymmetry of three-dimensional SQED. The real mass corresponds to a constant background vector field along the reduced direction.

When the masses of the tilded and untilded fields are equal, the renormalization of the FI term vanishes [14], and so do quantum corrections to the vortex mass. When we make the tilded fields heavy, $\tilde{m} \gg e\sqrt{\xi}$, effectively they become physical regulators. As long as we keep their mass \tilde{m} large but finite it acts as an ultraviolet cut-off in loop integrals. All one-loop corrections, including the linearly divergent part, become well-defined and perfectly transparent. We have a smooth transition as we eventually send \tilde{m} to infinity.

Our analysis is organized as follows. In Sect. 2 we describe our basic model obtained from four-dimensional SQED by reducing one of the spatial dimensions. We introduce the real mass \tilde{m} , to be treated as a free parameter, for the “second” chiral superfield. Section 3, carrying the main weight of this work, is devoted to quantum corrections to the central charge and vortex mass. The operator form of the central extension is discussed in detail in this section. In Sec. 4 we consider a global charge fractionalization and a related question of Chern–Simons.

2 Description of the model and classical results

Our starting point is $\mathcal{N} = 1$ SQED in 3+1 dimensions with two chiral matter superfields Φ and $\tilde{\Phi}$ and the Fayet–Iliopoulos term. It has four conserved supercharges. The corresponding Lagrangian is

$$\begin{aligned} \mathcal{L} = & \left\{ \frac{1}{4e^2} \int d^2\theta W_\alpha W^\alpha + \text{H.c.} \right\} + \int d^4\theta \Phi^* e^V \Phi \\ & + \int d^4\theta \tilde{\Phi}^* e^{-V} \tilde{\Phi} - \xi \int d^2\theta d^2\theta^\dagger V(x, \theta, \theta^\dagger), \end{aligned} \quad (4)$$

where W_α is the gauge field multiplet,

$$W_\alpha = \frac{1}{8} \bar{D}^2 D_\alpha V = \lambda_\alpha - \theta_\alpha D - i\theta^\beta F_{\alpha\beta} + i\theta^2 \partial_{\alpha\dot{\alpha}} \lambda^{\dagger\dot{\alpha}}. \quad (5)$$

In order to get $\mathcal{N} = 2$ supersymmetry in 2+1 dimensions we compactify one of the dimensions, say the third axis, keeping the zero Kaluza–Klein modes

and discarding nonzero ones. To introduce the tilded field mass we introduce a constant background gauge field along the compactified axis, V_{bg} , where the subscript bg means background. In terms of the components we have

$$V_{\text{bg}} = \theta^\dagger \gamma^0 \gamma^\mu \theta V_\mu^{\text{bg}}, \quad (6)$$

γ -matrices are defined in Eq. (31) below. The background vector field is chosen to be a constant field along the compactified axis, i.e. $V_{\text{bg}}^\mu = 2\tilde{m} \delta_3^\mu$. It is important to note that this is a new auxiliary field, rather than the expectation value of the original photon field. This background is coupled to $\tilde{\Phi}$ only, with the charge -1 . Then the Lagrangian takes the form

$$\begin{aligned} \mathcal{L} = & \left\{ \frac{1}{4e^2} \int d^2\theta W_\alpha W^\alpha + \text{H.c.} \right\} + \int d^4\theta \Phi^* e^V \Phi \\ & + \int d^4\theta \tilde{\Phi}^* e^{-V-V_{\text{bg}}} \tilde{\Phi} - \xi \int d^2\theta d^2\theta^\dagger V(x, \theta, \theta^\dagger), \end{aligned} \quad (7)$$

Upon introduction of the constant background field, $\tilde{\Phi}$ multiplet becomes massive whereas Φ multiplet is not affected, since it is chosen to be neutral with respect to the background field. It is clear that the kinetic term for the gauge multiplet is not affected, and similarly, the Fayet-Iliopoulos term remains the same since the superspace integral $\int d^4\theta V$ does not vanish only for the last component of the superfield V .

After compactification of the third axis, we get the following bosonic and fermionic Lagrangians in terms of the component fields (in the Wess–Zumino gauge):

$$\begin{aligned} \mathcal{L}_B = & -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}^\mu \tilde{\phi}^* \mathcal{D}_\mu \tilde{\phi} + \mathcal{D}^\mu \phi^* \mathcal{D}_\mu \phi + \frac{1}{2e^2} (\partial_\mu N)^2 \\ & + \frac{1}{2e^2} D^2 - \xi D + D(\phi^* \phi - \tilde{\phi}^* \tilde{\phi}) - N^2 \phi^* \phi - (\tilde{m} + N)^2 \tilde{\phi}^* \tilde{\phi}, \\ \mathcal{L}_F = & \frac{1}{e^2} \bar{\lambda} i \not{\partial} \lambda + \bar{\psi} i \not{\mathcal{D}} \psi + \bar{\tilde{\psi}} i \not{\mathcal{D}} \tilde{\psi} + N \bar{\psi} \psi - (\tilde{m} + N) \bar{\tilde{\psi}} \tilde{\psi} \\ & + i\sqrt{2} [(\bar{\lambda} \psi \phi^* - \bar{\psi} \lambda \phi)] - i\sqrt{2} [(\bar{\lambda} \tilde{\psi}) \tilde{\phi}^* - (\bar{\tilde{\psi}} \lambda) \tilde{\phi}], \end{aligned} \quad (8)$$

where $N = -A_3$ is a real pseudoscalar field, and

$$i\mathcal{D}_\mu \phi = (i\partial_\mu + A_\mu) \phi, \quad i\mathcal{D}_\mu \tilde{\phi} = (i\partial_\mu - A_\mu) \tilde{\phi}.$$

Moreover, D is an auxiliary field, which can be eliminated via its equation of motion. The Lagrangian (8) is invariant under the following supersymmetry transformations,

$$\begin{aligned}\delta\phi &= \sqrt{2}\bar{\epsilon}\psi, & \delta\psi &= \sqrt{2}(i\mathcal{D}\phi - eN\phi)\epsilon, \\ \delta\tilde{\phi} &= \sqrt{2}\bar{\epsilon}\tilde{\psi}, & \delta\tilde{\psi} &= \sqrt{2}\left(i\mathcal{D}\tilde{\phi} + e(N + \tilde{m})\tilde{\phi}\right)\epsilon, \\ \delta A_\mu &= i(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon), & \delta\lambda &= -\gamma^\mu\epsilon(\partial_\mu N - f_\mu) + i\epsilon\frac{D}{e},\end{aligned}\tag{9}$$

where

$$f_\mu = -\frac{i}{2}\epsilon_{\mu\alpha\beta}F^{\alpha\beta}, \quad D = e^2\left(|\phi|^2 - |\tilde{\phi}|^2 - \xi\right),$$

and $\epsilon = (\epsilon_1, \epsilon_2)$ is a complex spinor. The corresponding supersymmetry current is

$$\begin{aligned}j^\mu &= \sqrt{2}(\mathcal{D}\phi^* + ieN\phi^*)\gamma^\mu\psi + \sqrt{2}(\mathcal{D}\tilde{\phi}^* - ie(N + \tilde{m})\tilde{\phi}^*)\gamma^\mu\tilde{\psi} \\ &+ (i\mathcal{D}N - if + D)\gamma^\mu\lambda.\end{aligned}\tag{10}$$

The centrally extended algebra of the supercharges is discussed below, in Sect. 3.2, see Eq. (19). After elimination of the auxiliary D field via equation of motion, we get the following scalar potential:

$$V = \frac{e^2}{2}\left[\xi - (\phi^*\phi - \tilde{\phi}^*\tilde{\phi})\right]^2 + N^2\phi^*\phi + (\tilde{m} + N)^2\tilde{\phi}^*\tilde{\phi}.\tag{11}$$

If ξ is positive (and we will assume $\xi > 0$) the theory is in the Higgs regime and supports the BPS-saturated vortices. We will assume \tilde{m} to be positive too. If $\tilde{m} \neq 0$, the vacuum configuration is as follows:

$$\tilde{\phi} = 0, \quad N = 0, \quad \phi^*\phi = \xi.\tag{12}$$

Vortices with nonvanishing winding number correspond to windings of the ϕ field [15]. The fermionic fields are set to zero in the classical approximation.

We are interested in static solutions; the relevant part of the Lagrangian, upon the Bogomol'nyi completion [16], takes the form

$$\begin{aligned}\mathcal{L}_{\text{BPS}} &= -\frac{1}{2e^2}B^2 - |\mathcal{D}^i\phi|^2 - \frac{e^2}{2}\left[\xi - \phi^*\phi\right]^2 \\ &= -|\mathcal{D}_+\phi|^2 - \frac{1}{2e^2}\left[B - e^2(|\phi|^2 - \xi)\right]^2 \\ &\quad - \xi B - i\partial_k(\epsilon_{kl}\phi^*\mathcal{D}_l\phi),\end{aligned}\tag{13}$$

where $B = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field and $\mathcal{D}_+ \equiv \mathcal{D}_1 + i\mathcal{D}_2$

Since the solution is static we have $\mathcal{H} = -\mathcal{L}_{\text{BPS}}$. We will label the fields minimizing \mathcal{H} by the subscript (or superscript) v . They satisfy the following first-order BPS equations:

$$B_v - e^2 (|\phi_v|^2 - \xi) = 0, \quad \mathcal{D}_+^v \phi_v = 0. \quad (14)$$

The boundary conditions are self-evident. Solutions to these BPS equations in different homotopy classes are labeled by the winding number n . Needless to say, they are well known. A vortex with the winding number n has the mass

$$M_v = 2\pi n \xi, \quad (15)$$

where, at the classical level, the parameter ξ on the right-hand side is that entering the Lagrangian (8). At this level the central charge

$$|Z_v| = \xi \int d^2x B = 2\pi n \xi. \quad (16)$$

The vortex solution breaks 1/2 of supersymmetry. More precisely, the vortex solution is invariant under the supersymmetry transformations (9) restricted to $\epsilon = (0, \epsilon_2)$. In Sect. 3 we will show that this residual symmetry between bosons and fermions is strong enough to preserve the BPS saturation at the quantum level.

3 Quantum Corrections

In this section we will calculate quantum corrections to the Fayet–Iliopoulos parameter, the vortex mass and the central charge, using the regularization outlined in Sect. 1. We will keep \tilde{m} large but finite, taking the limit $\tilde{m} \rightarrow \infty$ at the very end. In order to calculate one-loop corrections to the classical results we will expand the fields around the background solutions

$$\phi = \phi_v + \eta, \quad A_\mu = A_\mu^v + a_\mu \quad (17)$$

keeping the terms quadratic in η , a_μ . The fields η and a_μ have the mass $m = e\sqrt{2\xi}$, while $\tilde{\phi}$ and $\tilde{\psi}$ have the mass \tilde{m} .²

²The superpartners ψ and λ do not have definite masses; the mass matrix for these fields can be diagonalized providing us with two diagonal combinations, $\psi' = \frac{\psi + i\lambda}{\sqrt{2}}$ and $\lambda' = \frac{\psi - i\lambda}{\sqrt{2}}$. The latter have masses $e\sqrt{2\xi}$. Note that both parameters, e and $\sqrt{\xi}$, have dimensions $[m]^{1/2}$.

3.1 Fayet–Iliopoulos parameter at one loop

As was mentioned, the Fayet–Iliopoulos parameter receives no corrections if $\tilde{m} = m$. If $\tilde{m} \neq m$, there is a one-loop quantum correction. The simplest way to compute the renormalization of ξ is to consider the Lagrangian before eliminating the auxiliary field D , i.e the bosonic part in Eq. (8). In this exercise we treat D as a constant background field. Figure 1 shows the tadpole diagrams arising from the couplings $D(\phi^*\phi - \tilde{\phi}^*\tilde{\phi})$, which renormalize ξ ,

$$\begin{aligned}\xi_{\text{R}} \equiv \xi + \delta\xi &= \xi + \int \frac{d^3k}{(2\pi)^3} \left(\frac{i}{k^2 - \tilde{m}^2} - \frac{i}{k^2 - m^2} \right) \\ &= \xi + \frac{m - \tilde{m}}{4\pi}.\end{aligned}\tag{18}$$

We see that \tilde{m} plays the role of the ultraviolet cut-off, as was expected. Needless to say, the finite part of the correction, $m/4\pi$, depends on the definition of the renormalized FI parameter. In fact, it has an infrared origin (otherwise, odd powers of m could not have entered). The renormalized FI parameter is defined as the coefficient in front of the D term in $\Gamma_{\text{one-loop}}$. Here we note a couple of differences between the result in Eq. (18) and the results in [3, 8]. The first difference is that \tilde{m} , which represents the linear divergence of ξ , is absent in the previous results since the authors used dimensional and zeta-function regularization, respectively. Another difference is the sign of the $\frac{m}{4\pi}$ term. The calculation of the vortex mass renormalization in [3, 8] was phrased as a counter term calculation; therefore, the result [3, 8] $\delta\xi = -\frac{m}{4\pi}$ which superficially has the sign opposite to that in Eq. (18) is in full accord with our result and with the central charge renormalization.

3.2 Central Charge

The nonvanishing (and linearly divergent) correction to ξ implies that the classical central charge in Eq. (16) must be corrected too, in accordance with Eq. (18), so that ξ is converted to ξ_{R} in the central charge. Now we will explain where this correction comes from.

The centrally extended superalgebra is

$$\{Q, (Q^\dagger) \gamma^0\} = 2(P_0 \gamma^0 + P_1 \gamma^1 + P_2 \gamma^2)$$

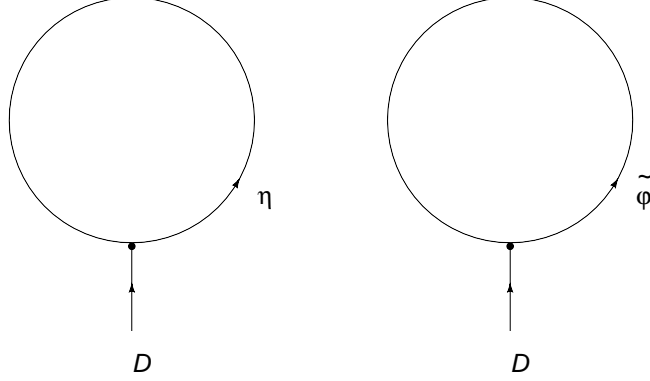


Figure 1: Tadpole diagrams determining one-loop correction to ξ .

$$- 2 \left(P_3 + \xi \int d^2x B \right), \quad (19)$$

where our conventions for the gamma matrices are summarized in Eq. (31) and P_3 is the “momentum” along the reduced direction,

$$P_3 = -\tilde{m} \int d^2x \left(i\tilde{\phi}^* \overleftrightarrow{\partial}_t \tilde{\phi} + \tilde{\psi} \gamma_0 \tilde{\psi} \right) \equiv \tilde{m} q. \quad (20)$$

Here q is the Noether charge of the vortex,

$$q = \int d^2x \tilde{J}_0, \quad \tilde{J}_\mu = - \left(i\tilde{\phi}^* \overleftrightarrow{\partial}_\mu \tilde{\phi} + \tilde{\psi} \gamma_\mu \tilde{\psi} \right). \quad (21)$$

The current \tilde{J}_μ defines a global U(1) symmetry acting in the regulator sector. Below we will show that the corresponding charge fractionalizes. (In the low-energy sector it is related to the occurrence of the Chern–Simons term after the tilded fermion is integrated out.)

It is rather obvious that the P_3 term is in one-to-one correspondence with the fact that integrating out massive fermions in $2 + 1$ dimensions generates the Chern–Simons term in the Lagrangian [10, 11], which, in turn, makes the vortex electrically charged [17]. Since our theory is fully regularized, the superalgebra (19) presents the exact operator equality in an explicit representation (which is sometimes elusive in other regularizations.) The second line in Eq. (19) is $-2 Z_v$. Although the coefficient of the Chern–Simons term in the Lagrangian is dimensionless, integrating out heavy fermions in the

central charge produces a term which has mass dimension 1. In fact, in Sect. 4 (see also Appendix) we will calculate the value of the Noether charge q (at one loop) and will show that $q = -\frac{n}{2}$. Note that for odd n the charge is fractional, a well known phenomenon of charge fractionalization [18].

Assembling two terms in the central charge and using the fact that $q = -\frac{n}{2}$ we get

$$\begin{aligned} |Z_{n,v}| &= 2\pi n \xi + \tilde{m} q = 2\pi n \xi - \frac{\tilde{m} n}{2} \\ &= n \left(2\pi \xi_R - \frac{m}{2} \right), \end{aligned} \quad (22)$$

where we used Eq. (18) to convert ξ into ξ_R . The contribution due to P_3 comes precisely in the combination ensuring that the bare parameter ξ is converted into the renormalized ξ_R . Equation (22) demonstrates the emergence of the quantum correction $-m n/2$.

3.3 Renormalization of the vortex mass

To calculate the one-loop contribution to the vortex mass, we expand the Lagrangian (8) around the background field, in the quadratic order, using the definitions (17). It is convenient to introduce the following gauge-fixing term:³

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2} \left(\frac{1}{e} \partial_\mu a^\mu + ie(\phi_v \eta^* - \phi_v^* \eta) \right)^2. \quad (23)$$

Note that under this gauge choice, a_0 becomes a dynamical field, and one has to take its loop contribution into account. The corresponding ghost Lagrangian is

$$\mathcal{L}_{\text{gh}} = \bar{c} \left[-\frac{1}{e^2} \partial_\mu \partial^\mu - (2|\phi_v|^2 + \phi_v \eta^* + \phi_v^* \eta) \right] c, \quad (24)$$

where \bar{c} and c are spin-zero complex fields with fermion statistics. We will drop the last two terms in Eq. (24) since they show up only in higher-order corrections. Assembling all the bosonic contributions, we get the following

³This gauge-fixing term is chosen to cancel the terms $(\eta^* \phi_v)^2$ and $(\eta \phi_v^*)^2$ originating from the scalar potential (11) as well as the term $\partial_\mu a^\mu (\eta^* \phi_v - \eta \phi_v^*)$ arising from the term $\mathcal{D}^\mu \phi^* \mathcal{D}_\mu \phi$ in Eq. (8).

bosonic Lagrangian (at the quadratic order)

$$\begin{aligned}
L_B^{(2)} &= \mathcal{L}_{\text{gf}}^{(2)} + \mathcal{L}_B^{(2)} + \mathcal{L}_{\text{gh}}^{(2)} \\
&= |\mathcal{D}_\mu^v \eta|^2 - e^2 (3|\phi_v|^2 - \xi) |\eta|^2 \\
&+ \frac{1}{2e^2} (\partial_\mu a_m)^2 - |\phi_v|^2 a_m^2 - 2ia^m (\eta^* \mathcal{D}_m^v \phi_v - \eta \mathcal{D}_m^v \phi_v^*) \\
&+ |\mathcal{D}_\mu^v \tilde{\phi}|^2 + \left[e^2 (|\phi_v|^2 - \xi^2) - \tilde{m}^2 \right] |\tilde{\phi}|^2 \\
&- \frac{1}{2e^2} (\partial_\mu a_0)^2 + |\phi_v|^2 a_0^2 + \frac{1}{2e^2} (\partial_\mu N)^2 - N^2 |\phi_v|^2 \\
&+ \bar{c} \left(-\frac{1}{e^2} \partial_\mu \partial^\mu - 2|\phi_v|^2 \right) c, \tag{25}
\end{aligned}$$

where $\mu = 0, 1, 2$ and $m = 1, 2$ (the fields η and a are defined in Eq. (17)). The last two lines in Eq. (25) include one complex scalar fields with the fermion statistics and two real scalar fields with the boson statistics, satisfying the same equations of motion. If we impose the same boundary conditions on the fields a_0 , N , \bar{c} and c , (and we do), they produce the same determinants, and their contributions to the vortex mass cancel each other [8]. With this observation in mind, we will drop this line in what follows.

The transverse components of the gauge field, a_1 and a_2 , can be combined into complex fields by defining

$$a^\pm = \frac{a^1 \pm ia^2}{\sqrt{2}e}. \tag{26}$$

By the same token, we define $\mathcal{D}_\pm^v = \mathcal{D}_1^v \pm i\mathcal{D}_2^v$. With these definitions Eq. (25) can be rewritten as follows:

$$\begin{aligned}
L_B^{(2)} &= |\mathcal{D}_\mu^v \eta|^2 - e^2 (3|\phi_v|^2 - \xi) |\eta|^2 \\
&+ \partial_\mu a^+ \partial^\mu a^- - 2e^2 |\phi_v|^2 a^+ a^- - \sqrt{2}ie (\eta^* a^+ \mathcal{D}_-^v \phi_v - \eta a^- \mathcal{D}_+^v \phi_v^*) \\
&+ |\mathcal{D}_\mu^v \tilde{\phi}|^2 + (e^2 (|\phi_v|^2 - \xi^2) - \tilde{m}^2) |\tilde{\phi}|^2. \tag{27}
\end{aligned}$$

Note that, at the quadratic order, the tilded bosonic sector is decoupled from the fluctuations of the nontilded one, i.e. $\tilde{\phi}$ is coupled to the background fields only. (We will soon see that the same decoupling occurs for the fermionic sector.) This allows us to consider the contributions of tilded and untilded fields separately.

3.3.1 One-loop contribution from the untilded sector

In the first part of this subsection we will compute the classical Hamiltonian (density) of the fluctuations. In the second part we will quantize the Hamiltonian by imposing canonical (anti)commutation relations. Finally we will compute the sum of the energies, which turns out to be vanishing. We first start with the bosonic Hamiltonian corresponding to the untilded part of the Lagrangian (27), which can be written in the matrix form,

$$\mathcal{H}_B^{(2)} = \begin{pmatrix} \dot{\eta}, & i\dot{a}_+ \end{pmatrix}^* \begin{pmatrix} \dot{\eta} \\ i\dot{a}_+ \end{pmatrix} + \begin{pmatrix} \eta, & i a_+ \end{pmatrix}^* D_B^2 \begin{pmatrix} \eta \\ i a_+ \end{pmatrix}, \quad (28)$$

where we defined the quadratic bosonic operator

$$D_B^2 = \begin{pmatrix} -(\mathcal{D}_k^v)^2 + e^2(3|\phi_v|^2 - \xi) & \sqrt{2}e\mathcal{D}_-^v\phi_v \\ \sqrt{2}e(\mathcal{D}_-^v\phi_v)^* & -\partial_k^2 + 2e^2|\phi_v|^2 \end{pmatrix}. \quad (29)$$

Eq. (28) gives the classical Hamiltonian for the bosonic fields. The fermionic Lagrangian (8) is already quadratic in the fermionic fields. Setting the bosonic fields to their background values gives the following quadratic Lagrangian for the untilded fermionic fields:

$$\mathcal{L}_F^{(2)} = \frac{1}{e^2} \bar{\lambda} i \not{\partial} \lambda + \bar{\psi} i \not{\mathcal{D}} \psi + i\sqrt{2} [(\bar{\lambda} \psi \phi^* - \bar{\psi} \lambda \phi)]. \quad (30)$$

We choose the following set of γ matrices:

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^2 = i\sigma_1. \quad (31)$$

With the chosen representation of γ matrices the Hamiltonian corresponding to the Lagrangian (30) reads

$$\begin{aligned} \mathcal{H}_F^{(2)} &= -i \begin{pmatrix} \psi_1 \\ \psi_2 \\ \lambda_1/e \\ \lambda_2/e \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathcal{D}_+^v & -\sqrt{2}e\phi_v & 0 \\ \mathcal{D}_-^v & 0 & 0 & \sqrt{2}e\phi_v \\ \sqrt{2}e\phi_v^* & 0 & 0 & \partial_+ \\ 0 & -\sqrt{2}e\phi_v^* & \partial_- & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \lambda_1/e \\ \lambda_2/e \end{pmatrix} \\ &= \begin{pmatrix} U \\ V \end{pmatrix}^\dagger \begin{pmatrix} 0 & -iD_F \\ iD_F^\dagger & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \end{aligned} \quad (32)$$

where we regrouped the components of λ and ψ ,

$$\{ U; V \} = \{ (\psi_1, \lambda_2/e); (\psi_2, \lambda_1/e) \}, \quad (33)$$

and defined the fermionic operator,

$$D_F \equiv \begin{pmatrix} \mathcal{D}_+^v & -\sqrt{2}e\phi_v \\ -\sqrt{2}e\phi_v^* & \partial_- \end{pmatrix}, \quad D_F^\dagger = \begin{pmatrix} -\mathcal{D}_-^v & -\sqrt{2}e\phi_v \\ -\sqrt{2}e\phi_v^* & -\partial_+ \end{pmatrix}. \quad (34)$$

Supersymmetry of the Lagrangian reveals itself when we calculate the following quadratic fermionic operator:

$$D_F^\dagger D_F = \begin{pmatrix} -(\mathcal{D}_k^v)^2 + e^2(3|\phi_v|^2 - \xi) & \sqrt{2}e\mathcal{D}_-^v\phi_v \\ \sqrt{2}e(\mathcal{D}_-^v\phi_v)^* & -\partial_k^2 + 2e^2|\phi_v|^2 \end{pmatrix}, \quad (35)$$

which coincides with D_B^2 defined in Eq. (29),

$$D_B^2 = D_F^\dagger D_F. \quad (36)$$

By virtue of this identification we rewrite the full Hamiltonian for untilded fields in terms of the operators D_F and D_F^\dagger ,

$$\begin{aligned} \mathcal{H}^{(2)} &= (\dot{\eta}, i\dot{a}_+)^* \begin{pmatrix} \dot{\eta} \\ i\dot{a}_+ \end{pmatrix} + (\eta, ia_+)^* D_F^\dagger D_F \begin{pmatrix} \eta \\ ia_+ \end{pmatrix} \\ &- iU^\dagger D_F V + iV^\dagger D_F^\dagger U. \end{aligned} \quad (37)$$

To quantize the Hamiltonian (37) we will follow methods worked out long ago (e.g. [19]). First, we impose boundary conditions which are compatible with the residual supersymmetry. We place the system into a spherical two-dimensional “box” of radius R , with the assumption that R is much larger than any length scale in the model at hand. To ensure that the energy associated with the boundary vanishes, we require all the fields to vanish at $r = R$. This condition does not break the residual supersymmetry since it is compatible with the transformations defined in Eq. (9). Then we expand the fields in Eq. (37) in eigenmodes of the operators D_B^2 and the associated operator $D_B^{2'}$ defined as follows:

$$D_B^{2'} = D_F D_F^\dagger. \quad (38)$$

The eigenvalue equations for these operators are

$$D_B^2 \xi_{n,\sigma} \equiv w_n^2 \xi_{n,\sigma}, \quad D_B^{2'} \xi'_{n,\sigma} \equiv w_n^2 \xi'_{n,\sigma}. \quad (39)$$

The eigenvalues for both operators are the same: the eigenfunctions can be related to each other by

$$\xi'_{n,\sigma} = \frac{1}{w_n} D_F \xi_{n,\sigma}, \quad \xi_{n,\sigma} = \frac{1}{w_n} D_F^\dagger \xi'_{n,\sigma}. \quad (40)$$

For each w_n^2 there are two independent solutions, which are labeled by subscript σ . The above statement excludes the zero modes, $w_n = 0$, which occur only in one of these operators, namely D_B^2 , reflecting the translational invariance in the problem at hand. Usually, they are referred to as translational. Their fermion counterparts, the zero modes of D_F , are supertranslational modes. D_F^\dagger has no zero modes.

The eigenfunctions $\xi_{n,\sigma}$ form an orthonormal and complete basis, in which we expand the fields in Eq. (37)

$$\begin{pmatrix} \eta(t, \mathbf{x}) \\ i a_+(t, \mathbf{x}) \end{pmatrix} = \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} a_{n,\sigma}(t) \xi_{n,\sigma}(\mathbf{x}),$$

$$V(t, \mathbf{x}) = \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} v_{n,\sigma}(t) \xi_{n,\sigma}(\mathbf{x}),$$

$$U(t, \mathbf{x}) = \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} u_{n,\sigma}(t) \xi'_{n,\sigma}(\mathbf{x}). \quad (41)$$

Note that the zero modes do not enter in the expansion (41), nor do they appear in the Hamiltonian (37). For nonzero modes the ratio of the bosonic to fermionic modes is 1:2, i.e. we have two complex expansion coefficients $a_{n,\sigma}(t)$ for bosons and four complex expansion coefficients $v_{n,\sigma}(t)$ and $u_{n,\sigma}(t)$ for fermions, for each value of w_n^2 . As we will see below, this is precisely what is needed for cancelation. Let us note in passing that for zero modes the ratio is 1:1. We have one complex bosonic modulus and one fermionic.

Using the above mode decompositions in Eq. (37), we arrive at an infinite set of oscillators,⁴

$$\begin{aligned} \mathcal{H}^{(2)} = & \sum_{\substack{n, n' \neq 0 \\ \sigma, \sigma'}} \left(\dot{a}_{n,\sigma}^* \dot{a}_{n',\sigma'} \xi_{n,\sigma}^\dagger \xi_{n',\sigma'} + a_{n,\sigma}^* a_{n',\sigma'} \xi_{n,\sigma}^\dagger D_F^\dagger D_F \xi_{n',\sigma'} \right. \\ & \left. - i \frac{1}{w_{n'}} v_{n,\sigma}^* u_{n',\sigma'} \xi_{n,\sigma}^\dagger D_F^\dagger D_F \xi_{n',\sigma'} + i w_n u_{n,\sigma}^* v_{n',\sigma'} \xi_{n,\sigma}^\dagger \xi_{n',\sigma'} \right). \end{aligned} \quad (42)$$

Now, for each oscillator, the coefficients a , \dot{a} , v , u and their complex conjugated must be represented as linear combinations of the corresponding creation and annihilation operator subject to the standard (anti)commutation relations. This procedure parallels that discussed in detail in Ref. [4]. The only difference is that in [4] for each mode one has an oscillator for one real degree of freedom, while in the case at hand we deal with a complex degree of freedom which is equivalent to two real degrees of freedom. We will not dwell on details referring the reader to Ref. [4]. Imposing the appropriate (anti)commutation relations on the creation and annihilation operators, we get for expectation values of bilinears in the vortex ground state

$$\begin{aligned} \langle a_{n,\sigma}^* a_{n',\sigma'} \rangle_{\text{vor}} &= \frac{1}{2w_n} \delta_{nn'} \delta_{\sigma\sigma'}, & \langle \dot{a}_{n,\sigma}^* \dot{a}_{n',\sigma'} \rangle_{\text{vor}} &= \frac{w_n}{2} \delta_{nn'} \delta_{\sigma\sigma'}, \\ \langle u_{n,\sigma}^* v_{n',\sigma'} \rangle_{\text{vor}} &= \frac{i}{2} \delta_{nn'} \delta_{\sigma\sigma'}, & \langle v_{n,\sigma}^* u_{n',\sigma'} \rangle_{\text{vor}} &= -\frac{i}{2} \delta_{nn'} \delta_{\sigma\sigma'}, \end{aligned} \quad (43)$$

where the angular brackets mark the vortex expectation value. Expectation values of all other bilinears vanish. If we substitute these results in Eq. (42) we immediately see that the one-loop correction in the untilded sector vanishes locally, i.e. in the Hamiltonian density. Needless to say, it vanishes in the integral $\int d^2x \mathcal{H}^{(2)}$ too.

Thus, we demonstrated the cancelation of the bosonic and fermionic contributions mode by mode, for each given n . This vanishing result shows that the vortex mass receives no correction from the untilded sector. If we did not have the $\tilde{\Phi}$ multiplet, this would be the final answer. However, the theory per se is ill-defined without the tilded sector.

⁴To be accurate we should note that here we use integration by parts in the last term, which means that Eq. (42) is valid up to a full spatial derivative.

From Eq. (18) we see that in the absence of $\tilde{\phi}$, the FI parameter ξ would be linearly divergent at one loop. With $\tilde{\phi}$ included, the theory is regularized; cancelation of loops in Fig. 1 takes place. The linear divergence is replaced by the linear dependence of ξ on \tilde{m} . The latter parameter is kept large, but finite till the very end. It is only natural that the linear dependence of $M_{v,R}$ on \tilde{m} will be provided by the tilded sector contribution (Sect. 3.3.2).

3.3.2 The tilded sector (regulator) contribution in M_v

The Lagrangian for the tilded sector is

$$\tilde{\mathcal{L}}^{(2)} = |\mathcal{D}_\mu^v \tilde{\phi}|^2 + \left(e^2 (|\phi_v|^2 - \xi^2) - \tilde{m}^2 \right) |\tilde{\phi}|^2 + \tilde{\psi} i \not{D} \tilde{\psi} - \tilde{m} \tilde{\psi} \tilde{\psi}. \quad (44)$$

The corresponding Hamiltonian density then takes the form

$$\begin{aligned} \tilde{\mathcal{H}}^{(2)} = & |\dot{\tilde{\phi}}|^2 + \tilde{\phi}^* (-\mathcal{D}_+^v \mathcal{D}_-^v + \tilde{m}^2) \tilde{\phi} \\ & + \begin{pmatrix} \tilde{\psi}_1 & \tilde{\psi}_2 \end{pmatrix}^* \begin{pmatrix} \tilde{m} & -i\mathcal{D}_+^v \\ -i\mathcal{D}_-^v & -\tilde{m} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}, \end{aligned} \quad (45)$$

where $\mathcal{D}_\pm^v = \mathcal{D}_1^v \pm i\mathcal{D}_2^v$ and we used Eq. (14). For what follows it is important to know that the operator $\mathcal{D}_+^v \mathcal{D}_-^v$ has no zero modes.

If we denote the eigenvalues of the bosonic operator

$$-\mathcal{D}_+^v \mathcal{D}_-^v + \tilde{m}^2 \quad (46)$$

by Δ (Δ is strictly larger than \tilde{m}^2), for each given Δ we have two eigenmodes of the associated fermion equation

$$\begin{pmatrix} \tilde{m} & -i\mathcal{D}_+^v \\ -i\mathcal{D}_-^v & -\tilde{m} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = \pm \sqrt{\Delta} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}. \quad (47)$$

The eigenfunctions have the following structure. If $\tilde{\psi}_1$ is the normalized eigenfunction of the operator $-\mathcal{D}_+^v \mathcal{D}_-^v$, then $\tilde{\psi}_2$ is the corresponding eigenfunction of the conjugated operator $-\mathcal{D}_-^v \mathcal{D}_+^v$ times

$$\sqrt{\frac{\pm\sqrt{\Delta} - \tilde{m}}{\pm\sqrt{\Delta} + \tilde{m}}}$$

depending on the sign in the eigenvalue equation (47). Thus, for each complex boson mode with the eigenvalue Δ we have two complex fermion modes with the eigenvalues $\pm\sqrt{\Delta}$. This balance of modes guarantees that the corresponding quantum corrections to M_v vanish.

This is not the end of the story, however. There is *one* additional (complex) fermion mode with Δ exactly equal to \tilde{m}^2 . (The above statement refers to the elementary vortex with the unit winding number. Generalization to higher winding numbers is straightforward.) Let us focus on this unbalanced mode which will be solely responsible for the contribution of the tilded sector in M_v .

From Eqs. (47) and (14) it is clear that this fermion mode has the form

$$\begin{pmatrix} 0 \\ \tilde{\psi}_2^{(0)} \end{pmatrix}, \quad (48)$$

where the eigenvalue on the right-hand side of Eq. (47) is $-\tilde{m}$. This gives rise to the following contribution in the energy density:

$$\mathcal{E}^{(0)} = -\tilde{m} (\tilde{\psi}_2^{(0)})^* \tilde{\psi}_2^{(0)}. \quad (49)$$

We proceed to quantization in the standard manner. To this end we represent

$$\tilde{\psi}_2^{(0)} = \alpha^\dagger(t) \varphi(\mathbf{x}), \quad (50)$$

where $\varphi(\mathbf{x})$ is the normalized *c*-numerical part of the zero mode while α^\dagger is the operator part with the appropriate anticommutation relation implying

$$\langle \alpha \alpha^\dagger \rangle = \frac{1}{2}. \quad (51)$$

Now, the contribution of the tilded sector to M_v obviously reduces to

$$\delta M \equiv \int d^2x \langle \mathcal{E}^{(0)} \rangle = -\tilde{m} \langle \alpha \alpha^\dagger \rangle = -\frac{\tilde{m}}{2}. \quad (52)$$

Equation (52) gives the only nonvanishing quantum correction,

$$M_R \equiv M + \delta M = 2\pi \xi - \frac{\tilde{m}}{2} = 2\pi \xi_R - \frac{m}{2}, \quad (53)$$

where we again used Eq. (18) to convert ξ to ξ_R . Comparing this result with the renormalization of the central charge in Eq. (22), we conclude that the BPS saturation does indeed hold at the quantum level.

3.4 Higher orders

Let us discuss now what changes as we pass to higher orders of perturbation theory. Returning to Sect. 3.2 and, in particular, to Eq. (20), it is not difficult to understand that the relation $Z = 2\pi\xi - \frac{1}{2}\tilde{m}$ (for the elementary vortex) remains *exact* to all orders. Indeed, q is half-integer and the relation $q = -\frac{1}{2}$ for the elementary vortex cannot receive corrections⁵ in $e/\sqrt{\xi}$. If we define $\tilde{\xi}$ as

$$\tilde{\xi} = \xi - \frac{\tilde{m}}{4\pi}, \quad (54)$$

where ξ and \tilde{m} are bare parameters, then the statement that

$$M_v = Z = 2\pi\tilde{\xi} \quad (55)$$

is valid to all orders. The term \tilde{m} comes from the ultraviolet, and, therefore, it is natural to refer to $\tilde{\xi}$ as to an “effective ultraviolet parameter.” Equation (55) is akin to the NSVZ theorem for the gauge coupling renormalization in four dimensions [21]: being expressed of terms of the ultraviolet (bare) parameters the gauge coupling renormalization is limited to one loop (see also the second paper in [14]).

Corrections in powers of $e/\sqrt{\xi}$ arise if we decide to express the result in terms of ξ_R , a parameter defined in the infrared; the expression of ξ_R in terms of ξ does contain an infrared contribution (otherwise, odd powers of e could not have entered, see Sect. 3.1). Generalizing the arguments of [14] we can write, instead of (18)

$$\tilde{\xi} = \xi_R \left(1 - \frac{1}{2\sqrt{2}\pi} \frac{e}{\sqrt{\xi_R}} \right). \quad (56)$$

Equations (55) and (56) assembled together present a perturbatively exact result for $M_v = Z$.

4 Calculation of the Noether charge q

In Sect. 3.2 we used the fact that the Noether U(1) charge of the elementary vortex is $-1/2$. The Noether charge is saturated by the fermion term in

⁵A simple dimensional analysis shows that perturbative corrections run in powers of $e/\sqrt{\xi}$.

Eq. (20),

$$q = - \int d^2x \tilde{\psi} \gamma^0 \psi. \quad (57)$$

Here we will explore this issue in more detail. The vortex Noether charge can be calculated in a number of ways. The most straightforward calculation is that of the Feynman diagram depicted in Fig. 2, using the background field expansion. This expansion is justified because the background photon field is small compared to the value of \tilde{m} (in the very end we want to tend \tilde{m} to infinity). For our purposes it is sufficient to limit ourselves to the leading

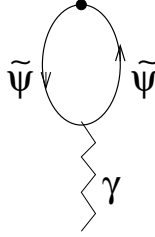


Figure 2: Calculation of the vortex Noether charge. We trace the term linear in the momentum k of the produced photon (assuming k to be small). Terms of the zeroth order vanish because of the gauge invariance. Quadratic and higher order terms in k are irrelevant since they are suppressed by powers of $1/\tilde{m}$.

term (proportional to $F_{\alpha\beta}$). Using γ^μ in the upper vertex in Fig. 2 (denoted by the closed circle) we get the Noether current in the background field in the form

$$\tilde{\psi} \gamma^\mu \psi \longrightarrow \tilde{m} F_{\alpha\beta} \epsilon^{\mu\alpha\beta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2 + \tilde{m}^2)^2} = \frac{1}{8\pi} F_{\alpha\beta} \epsilon^{\mu\alpha\beta}. \quad (58)$$

The current in (58) couples to the gauge field A_μ , giving a term of the form $A_\mu F_{\alpha\beta} \epsilon^{\mu\alpha\beta}$, which is nothing but the Chern–Simons term. Now, if we set $\mu = 0$ in (58) and invoke the standard value of the magnetic flux,

$$\int d^2x B = 2\pi,$$

we immediately get

$$\langle q \rangle = -1/2. \quad (59)$$

5 Conclusion

In this paper we showed that the mass and the central charge of the $\mathcal{N} = 2$ vortices in $2 + 1$ dimensions, being expressed in terms of ξ_R , get a quantum correction $-m n/2$ where m is the mass of the charged bosons (fermions) and n is the winding number of the vortex. The equality of the corrections to the vortex mass/central charge shows that the BPS saturation persists at the quantum level. Our result is in agreement with the previous ones [8, 6].

New elements of our work (compared to [8] and [6]) are as follows. We use a more straightforward and physically transparent regularization scheme which captures linearly divergent terms invisible in the regularization methods used in the previous papers. In our scheme we have a massive regulator multiplet acting in loops as an ultraviolet cutoff. In the limit of infinitely large regulator mass, regulator's role is taken over by the Chern–Simons term. We establish a contact between one-loop calculations and the general operator expression for the central charge (obtained within the same regularization scheme). Analyzing both, in a single package, we are able to reveal a simple physical interpretation behind the occurrence of the $-m n/2$ shift, and obtain all-order results (Sect. 3.4).

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Appendix

It is instructive to illustrate calculations of the charge q by inspecting the fermion mode decomposition discussed in Sect. 3.3.2. It is important to note that the mode decomposition in Sect. 3.3.2 is *not* the canonical expansion. A similar charge calculation by virtue of the canonical expansion was first performed in [23]. We will discuss both methods.

First, we expand the fields $\tilde{\psi}_1$ and $\tilde{\psi}_2$ in terms of the eigenfunctions of the operators $-\mathcal{D}_+^v \mathcal{D}_-^v$ and $-\mathcal{D}_-^v \mathcal{D}_+^v$, namely, in $\eta_{n,\sigma}$ and $\eta'_{n,\sigma}$, respectively:

$$\begin{aligned}\tilde{\psi}_1 &= \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} v_{n,\sigma}(t) \eta_{n,\sigma}(\mathbf{x}), \\ \tilde{\psi}_2 &= \tilde{\psi}_2^{(0)} + \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} u_{n,\sigma}(t) \eta'_{n,\sigma}(\mathbf{x}),\end{aligned}\tag{A.1}$$

where σ labels two independent solutions corresponding to the same eigenvalue, and $\tilde{\psi}_2^{(0)}$ is the zero mode defined in Eq. (50). The nonvanishing bilinears constructed from $u_{n,\sigma}(t)$ and $v_{n,\sigma}(t)$ are given in Eq. (43). With the expansion in Eq. (A.1) in hands, it is easy to see that the only nonvanishing contribution to q comes from the zero mode of the operator \mathcal{D}_+^v . This statement is a consequence of the following expansion of q :

$$\begin{aligned}\langle q \rangle &= - \int d^2x \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \\ &= - \langle \alpha \alpha^\dagger \rangle - \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} \langle u_{n,\sigma}^* u_{n,\sigma} + v_{n,\sigma}^* v_{n,\sigma} \rangle.\end{aligned}\tag{A.2}$$

Using Eqs. (43) and (51) we get

$$\langle q \rangle = -1/2,\tag{A.3}$$

in perfect agreement with the previous result (59).

(The fact that $q = -1/2$ on the vortex is in one-to-one correspondence with the fact that integrating out the massive fermion $\tilde{\psi}$ we generate the Chern–Simons term with $\kappa = \frac{e}{4\pi}$ [11]. It is well known that selfdual n -vortices with the Chern–Simons term have charge $q = -\frac{2\pi n\kappa}{e} = -\frac{n}{2}$ where n is the winding number [22, 17].)

We can carry out a slightly different calculation of the q charge by expanding the tilded fermion field in the canonical basis. However, we should remember that, generally speaking, the U(1) charge of the vacuum is infinite

in the absence of proper regularization.⁶ The same “vacuum” infinity then shows up in q . In fact, we are interested in the difference between the values of q on the vortex and in the vacuum.

This problem is automatically solved if, instead of the charge $-\int d^2x \tilde{\psi}^\dagger \tilde{\psi}$, one uses the following definition:

$$q = -\frac{1}{2} \int d^2x \left(\tilde{\psi}^\dagger \tilde{\psi} - \tilde{\psi}_c^\dagger \tilde{\psi}_c \right), \quad (\text{A.4})$$

where $\tilde{\psi}_c = -i(\tilde{\psi}^\dagger \gamma_2)^T$ is the charge-conjugated fermion field. We now expand the fermionic field $\tilde{\psi}$ in the canonical basis,

$$\tilde{\psi} = a_0^\dagger \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix} + \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} \left(e^{-i w_n t} \frac{a_{n,\sigma}}{\sqrt{2}} \varphi_{n,\sigma} + e^{i w_n t} \frac{b_{n,\sigma}^\dagger}{\sqrt{2}} \varphi_{n,\sigma}^* \right), \quad (\text{A.5})$$

where $\varphi_{n,\sigma}$ are the energy eigenfunctions of the fermionic Hamiltonian with the eigenvalues w_n . The operators a_0 , $a_{n,\sigma}$ and $b_{n,\sigma}$ obey the canonical anti-commutation relations⁷

$$\{a_0, a_0^\dagger\} = 1, \quad \{a_{n,\sigma}, a_{n',\sigma'}^\dagger\} = \delta_{n,n'} \delta_{\sigma,\sigma'}, \quad \{b_{n,\sigma}, b_{n',\sigma'}^\dagger\} = \delta_{n,n'} \delta_{\sigma,\sigma'}. \quad (\text{A.6})$$

The operators $a_{n,\sigma}$ and $b_{n,\sigma}^\dagger$ are the annihilation and creation operators associated with the positive and negative energy solutions.⁸ The first term in the expansion (A.5) is the zero mode. Inserting the expansion (A.5) into Eq. (A.4), we get

$$\begin{aligned} \langle q \rangle &= -\frac{1}{2} \langle a_0 a_0^\dagger - a_0^\dagger a_0 \rangle \\ &- \sum_{\substack{n \neq 0 \\ \sigma = 1, 2}} \langle a_{n,\sigma}^\dagger a_{n,\sigma} - b_{n,\sigma}^\dagger b_{n,\sigma} - a_{n,\sigma} a_{n,\sigma}^\dagger + b_{n,\sigma} b_{n,\sigma}^\dagger \rangle \end{aligned} \quad (\text{A.7})$$

⁶This infinity does not show up in Eq. (A.3) because a regularized definition (A.4) of the q charge is built in in the expansion coefficients.

⁷Needless to say, all other anticommutators, not indicated in (A.6), vanish.

⁸The operators a_0 and a_0^\dagger are not necessarily required to be particle annihilation and creation operators, see Ref. [23] for details.

The condition we impose on a is $a|\text{vor}\rangle = 0$. With this condition we get

$$\langle q \rangle = -1/2, \tag{A.8}$$

which again agrees with the previous results.

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